Continuous Random Variable III

Aug 4, 2022

Joint distribution: Joint PDF

- A joint density function for two continuous random variables X, Y is a function $f: \mathbb{R}^2 \to \mathbb{R}$, such that
  - f is nonnegative, $f_{X,Y}(x, y) \geq 0, \forall x, y \in \mathbb{R}$
  - Total integral is 1, $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dx\, dy = 1$

- The joint distribution of two continuous random variables X, Y is given by, $\forall a \leq b, c \leq d$

$$\mathbb{P}(a \leq X \leq b, c \leq Y \leq d) = \int_c^d \int_a^b f_{X,Y}(x, y) dx\, dy\ .$$

Normal random variable

(normal distribution, Gaussian distribution)

- A continuous random variable $\boldsymbol{X}$ is normal or Gaussian if the PDF is in the form

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\,\sigma^2}}$$

- $\mathbb{E}(X) = \mu, Var(X) = \sigma^2$

- $\boldsymbol{X} \sim \mathcal{N}(\mu, \sigma^2)$

Useful integral $\int_{-\infty}^{\infty} e^{-x^\wedge 2}\, dx = \sqrt{\pi}$

Normal random variable

(normal distribution, Gaussian distribution)

- A continuous random variable $\boldsymbol{X} \sim \mathcal{N}(\mu, \sigma^2), a, b \neq 0, \boldsymbol{Y} = a\boldsymbol{X} + b$. Then $\boldsymbol{Y} \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$

- Further if $\boldsymbol{Y} = \frac{\boldsymbol{X} - \mu}{\sigma}$, then $\boldsymbol{Y} \sim \mathcal{N}(0,1)$

Sum of i.i.d. Normal

- Let $X \sim \mathcal{N}(0,1), Y \sim \mathcal{N}(0,1), X \perp Y$. Let $a, b \in \mathbb{R}$ be constant. Then $Z = aX + bY \sim \mathcal{N}(0, a^2 + b^2)$

- A general case, let $X \sim \mathcal{N}(\mu_1, \sigma_1^2), Y \sim \mathcal{N}(\mu_2, \sigma_2^2), X \perp Y$. Let $a, b \in \mathbb{R}$ be constant. Then $Z = aX + bY \sim \mathcal{N}(a\mu_1 + b\mu_2, a^2\sigma_1^2 + b^2\sigma_2^2)$

CDF of standard normal

- CDF of $\mathcal{N}(0,1)$ standard normal is denote by Φ

$$\Phi(y) = \mathbb{P}(Y \le y) = \mathbb{P}(Y < y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{t^2}{2}}\, dt$$

- CDF for $\boldsymbol{X} \sim \mathcal{N}(\mu, \sigma^2)$ calculation
  1. standardize X by defining a new normal r.v. $Y = \frac{\boldsymbol{X}-\mu}{\sigma}$
  2. $\mathbb{P}(X \le x) = \mathbb{P}\left(\frac{\boldsymbol{X}-\mu}{\sigma} \le \frac{x-\mu}{\sigma}\right) = \mathbb{P}\left(Y \le \frac{x-\mu}{\sigma}\right) = \Phi(\frac{x-\mu}{\sigma})$

Sample mean: $M_n = \frac{S_n}{n}$

Central Limit Theorem (CLT)

Let $X_1, X_2, \dots, X_n$ be a sequence of iid random variables with $\mathbb{E}(X_i) = \mu,\ Var(X_i) = \sigma^2$

$$Z_n = \frac{S_n - n\mu}{\sigma\sqrt{n}} = \frac{X_1 + X_2 + \cdots + X_n - n\mu}{\sigma\sqrt{n}}$$

$$\mathbb{E}(Z_n) = 0, Var(Z_n) = \frac{n\sigma^2}{\sigma^2 n} = 1$$

The CDF of Z_n converge to standard normal CDF

$$\lim_{n\to\infty} \mathbb{P}(Z_n \le z) = \Phi(z), \forall z$$

Normal approximation based on CLT

Let $X_1, X_2, \dots, X_n$ be a sequence of iid random variables with $\mathbb{E}(X_i) = \mu,\ Var(X_i) = \sigma^2$. If n is large, $\mathbb{P}(S_n \le c)$ can be approximated by treating S_n as if it were normal:

1. Calculate the mean nμ and the variance $n\sigma^2$ of S_n
2. calculate the normalization value $z = \frac{c - n\mu}{\sigma\sqrt{n}}$ (z-score)
3. Use approximation $\mathbb{P}(S_n \le c) \approx \Phi(z)$

   where $\Phi(z)$ is available from standard normal CDF table.

Example 1. Polling

We want to find out the value p representing the fraction of people supporting candidate A in a city. How many people we need to interview if we wish to estimate within accuracy of 0.01 with 95% probability.

Example 2.

We load on a plane 100 packages, weight of each package is independent random variable follows uniform distribution between 5-50 kg. What is the probability that the total weight will exceed 3000 kg?

Conditioning on an event

Conditional PDF of a continuous random variable X, given an event A with $\mathbb{P}(A) > 0$, is defined as a nonnegative function $f_{X|A}$ that satisfies

$$\mathbb{P}(X \in B|A) = \int_B f_{X|A}(x)dx$$

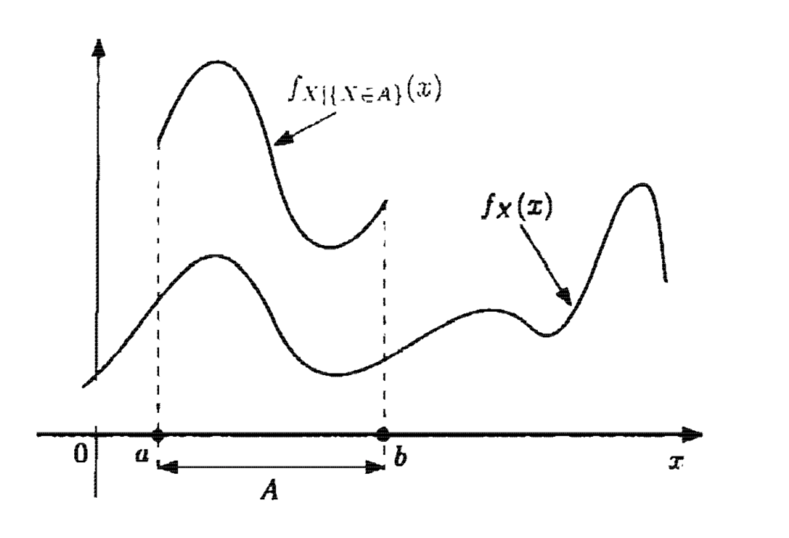
for any subset B of the real line.

Conditioning on an event

- If the event we condition on takes form of $\{X \in A\}$ and $\mathbb{P}(X \in A) > 0$ then

$$\mathbb{P}(X \in B | X \in A) = \frac{\mathbb{P}(X\in B, X\in A)}{\mathbb{P}(X\in A)} = \frac{\int_{A\cap B} f_X(x)dx}{\mathbb{P}(X\in A)}$$

- And $f_{X|\{X\in A\}}(x) = \begin{cases} \frac{f_X(x)}{\mathbb{P}(X\in A)} & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$



Example 3. Exponential is memoryless

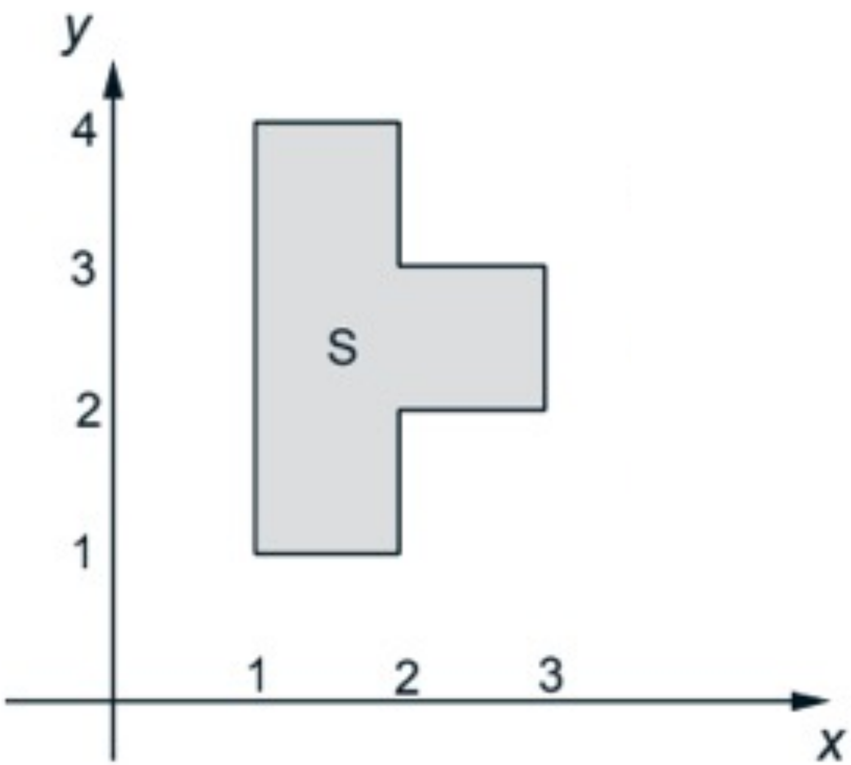
The time T till a new light bulb burns out is an exponential random variable with parameter λ. Alice turns the light on, leaves the room and when she returns, t time later, finds that the light bulb is still on, which correspond to event $A = \{T > t\}$. Let X be the additional time till the light bulb burns out. What is the conditional CDF of X. Given the event A?

Conditioning on another random variable

- Two random variables X, Y with joint PDF $f_{X,Y}$. For any fixed y with $f_Y(y) > 0$ the conditional PDF of X given $Y = y$ is defined by
$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

Example 4.

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$



$$f_{X|Y}(x|Y = 3.5)$$

$$f_{X|Y}(x|Y = 2.5)$$

$$f_{X|Y}(x|Y = 1.5)$$

Conditioning on another random variable

- Two random variables X, Y with joint PDF $f_{X,Y}$. The joint, marginal and conditional PDFs are

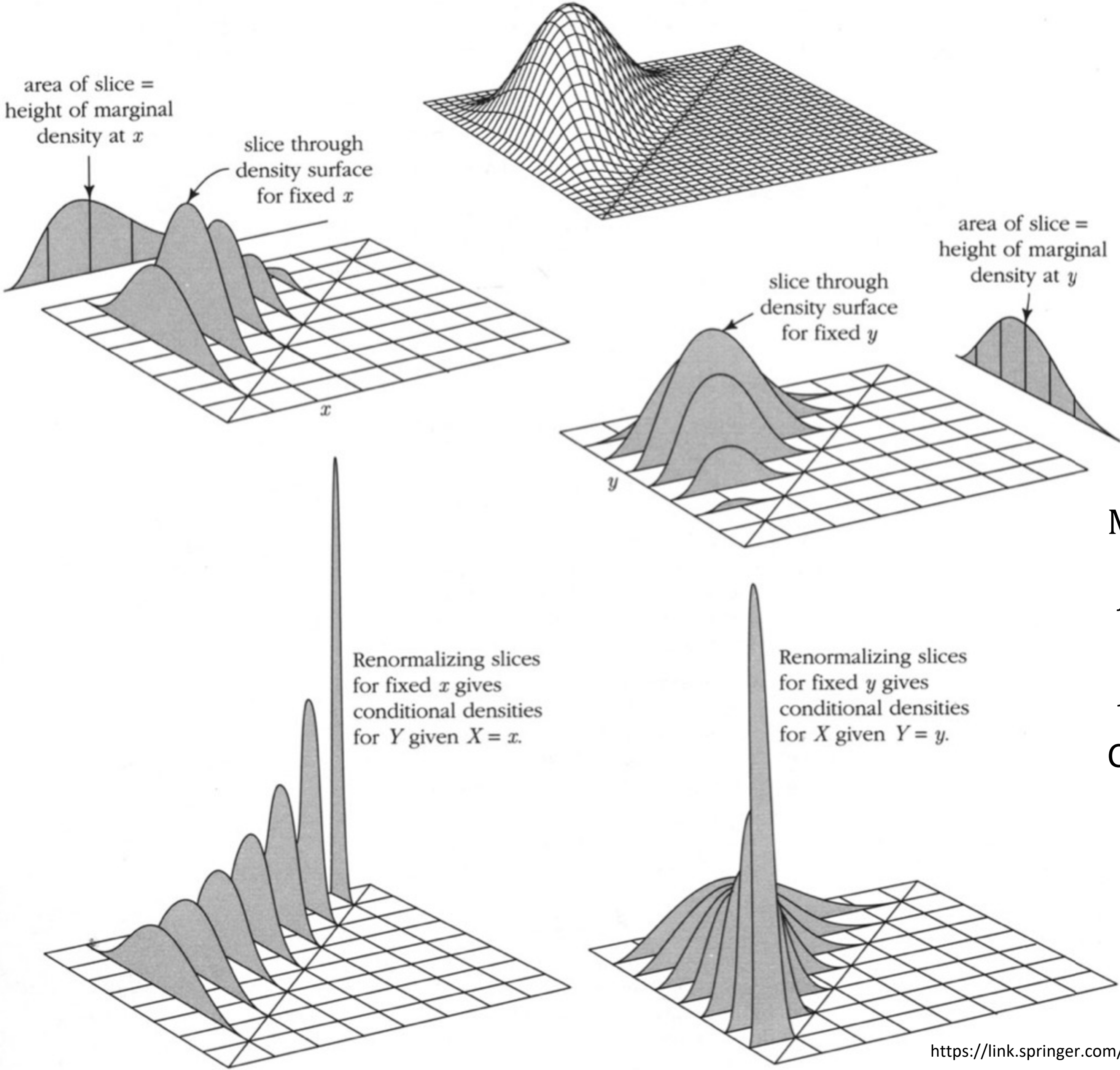
$$f_{X,Y}(x, y) = f_{X|Y}(x|y) f_Y(y)$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{X|Y}(x|y) f_Y(y)\, dy$$

The conditional PDF $f_{X|Y}(x|y)$ is defined for those y for which $f_Y(y) > 0$

- $f_{X|Y}(x|y)$ is a legit PDF, we can use it to calculate probability

$$\mathbb{P}(X \in \mathrm{A}|Y = y) = \int_A f_{X|Y}(x|y) dx$$



Marginals:

$$f_{\boldsymbol{X}}(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx$$

Conditional:

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

https://link.springer.com/content/pdf/10.1007%2F978-1-4612-4374-8_6.pdf

Conditional expectation

- Let X, Y be jointly continuous random variable, and let A be an event with $\mathbb{P}(A) > 0$

The conditional expectation of X given the event A is

$$\mathbb{E}(X|A) = \int_{-\infty}^{\infty} x\, f_{X|A}(x)dx$$

The conditional expectation of X given $Y = y$ is

$$\mathbb{E}(X|Y = y) = \int_{-\infty}^{\infty} x\, f_{X|Y}(x|y)dx$$

Law of iterative expectation

Let $A_1, A_2, \dots, A_n$ be disjoint events that form a partition of the sample space, and assume that $\mathbb{P}(A_i) > 0, \forall i$,

$$\mathbb{E}(X) = \sum_{i=1}^{n} \mathbb{P}(A_i)\mathbb{E}(X|A_i)$$

Similarly

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} \mathbb{E}(X|Y = y) f_Y(y)\, dy$$

Overall,

$$\mathbb{E}(X) = \mathbb{E}(\mathbb{E}(X|Y))$$

Example 5. Mean and variance of a piecewise constant PDF

Consider a random variable X has piecewise constant PDF

- $f_X(x) = \begin{cases} \frac{1}{3} & \text{if } 0 \le x \le 1 \\ \frac{2}{3} & \text{if } 1 < x \le 2 \\ 0 & \text{otherwise} \end{cases}$

consider events

- $A_1 = \{X \text{ lies in the first interval } [0,1]\}$
- $A_2 = \{X \text{ lies in the second interval } (1,2]\}$